M337/Practice exam 1

Module Examination

Complex analysis

Time allowed: 3 hours

There are **two parts** to this examination.

In **Part 1** you should **submit answers to <u>all</u> 6 questions**. Each question is worth 10% of the total mark.

In **Part 2** you should **submit answers to <u>2 out of the 3</u> questions**. Each question is worth 20% of the total mark.

Do not submit more than two answers for Part 2. If you submit answers to all three Part 2 questions, then only Questions 7 and 8 will be marked.

Include all your working, as some marks are awarded for this.

Write your answers in **pen**, though you may draw diagrams in pencil.

Start your answer to each question on a new page, clearly indicating the number of the question.

Crossed out work will not be marked.

Follow the instructions in the online timed examination for how to submit your work. Further information about completing and submitting your examination work is in the *Instructions and guidance for your remote examination* document on the module website.

1.1

Part 1

You should **submit answers to all questions** from Part 1.

Each question is worth 10%.

Question 1

Let $\alpha = 3 - 3i\sqrt{3}$.

(a) Determine each of the following complex numbers in *Cartesian* form, simplifying your answers as far as possible.

(i) $\dfrac{3}{\alpha}$ *[2]*

(ii) α^2 *[2]*

(iii) $\operatorname{Log} \alpha^2$ *[3]*

(b) Find all the fourth roots of α in *polar* form. *[3]*

Question 2

(a) Let Γ be the smooth path

$$\Gamma : \gamma(t) = 2e^{it} \quad (t \in [-\pi/2, 0]).$$

(i) Evaluate

$$\int_\Gamma \overline{z}\, dz.$$ *[3]*

(ii) Using your answer to part (a)(i), or otherwise, evaluate

$$\int_{\widetilde{\Gamma}} \overline{iz}\, dz,$$ *[2]*

where $\widetilde{\Gamma}$ is the reverse path of Γ.

(b) Determine an upper estimate for the modulus of

$$\int_C \frac{7\cosh z}{z^4 + 12}\, dz,$$

where C is the circle $\{z : |z| = 3\}$. *[5]*

Question 3

(a) Find the residue of the function

$$f(z) = \frac{z+4}{z^3 + z}$$

at each of its poles. *[4]*

(b) Hence evaluate the real improper integrals

$$\int_{-\infty}^{\infty} \left(\frac{t+4}{t^3+t}\right) \cos 2t\, dt \quad \text{and} \quad \int_{-\infty}^{\infty} \left(\frac{t+4}{t^3+t}\right) \sin 2t\, dt.$$ *[6]*

Question 4

Let f be the Möbius transformation

$$f(z) = \frac{z+i}{iz+2},$$

and let $C = \{z : |z-i| = 2\}$.

(a) Find the point β such that $\alpha = \infty$ and β are inverse points with respect to C. [2]

(b) Determine an equation for the image circle $f(C)$ in Apollonian form. [5]

(c) Find the centre and radius of $f(C)$. [3]

Question 5

(a) Let q be the velocity function

$$q(z) = \frac{i\,\overline{z}^2}{\overline{z}+i}$$

for an ideal flow with flow region $\mathbb{C} - \{i\}$. Use the Circulation and Flux Contour Integral to classify i as a source, sink or vortex of the flow. [5]

(b) Let $\mathcal{R} = \{z : \operatorname{Re} z < 0\} - \{-1\}$. Use the Joukowski function to find a one-to-one conformal mapping from $\mathcal{R}$ onto $\mathbb{C} - [-2, 2]$. [5]

Question 6

(a) Prove that the iteration sequence

$$z_{n+1} = (iz_n - 1)(z_n + 2i), \quad n = 0, 1, 2, \ldots,$$

with $z_0 = 0$, is conjugate to the iteration sequence

$$w_{n+1} = w_n^2 - \tfrac{7}{4}, \quad n = 0, 1, 2, \ldots,$$

with $w_0 = -\frac{3}{2}$.

State the conjugating function. [4]

(b) Determine whether or not each of the following points c lies in the Mandelbrot set.

(i) $c = -1 + \frac{1}{5}i$ [3]

(ii) $c = \frac{1}{5} + \frac{2}{5}i$ [3]

Part 2

You should **submit answers to two questions** from Part 2. If you submit answers to all three Part 2 questions, then only Questions 7 and 8 will be marked.

Each question is worth 20%.

Question 7

(a) Let

$$A = \{z : -2 \leq \operatorname{Re} z \leq 2,\ -3 \leq \operatorname{Im} z \leq 3\},$$
$$B = \{x + 2i : x > 0\},$$
$$C = \mathbb{C} - B.$$

(i) Sketch the sets $A \cap B$ and $A \cap C$. *[4]*

(ii) Determine whether or not the function $f(z) = \sin(z - 2i)$ is bounded on each of the sets A, B and C. *[6]*

(b) Use the Cauchy–Riemann Theorem and its converse to prove that the function

$$f(z) = (\operatorname{Re} z) \exp(\overline{z})$$

is differentiable at $z = -\frac{1}{2} + iy$ for any $y \in \mathbb{R}$ and nowhere else.

Hence find the derivative of f at the point $z = -\frac{1}{2}$. *[10]*

Question 8

(a) Find the radius of convergence of each of the following power series.

(i) $\displaystyle\sum_{n=0}^{\infty} \frac{i^n(n+1)!}{5^n}(z-3)^n$ *[3]*

(ii) $\displaystyle\sum_{n=0}^{\infty}(3n^2 - 5n + e^{in})(z-3i)^n$ *[3]*

(b) (i) Find the Taylor series about 0 for the function

$$g(z) = \cos(\sinh z),$$

up to the term in z^4.

Explain why the series represents g on $\mathbb{C}$. *[5]*

(ii) Hence evaluate the integral

$$\int_C z\, g(3/z)\, dz,$$

where C is the circle $\{z : |z| = 1\}$. *[4]*

(c) Suppose that f is an entire function that satisfies

$$f\left(\frac{1}{3^n}\right) = \frac{1}{3^{n+1}},$$

for all $n \in \mathbb{Z}$.

Find an entire function that satisfies this property, and prove that it is in fact the only entire function with this property. *[5]*

Question 9

(a) (i) Use the Taylor series about 0 for sin and sinh to prove that

$$|\sin z| < \tfrac{3}{2}, \quad \text{for } |z| = 1.$$ *[4]*

(ii) Use your answer to part (a)(i) and Rouché's Theorem to find all the solutions of the equation

$$2z + \sin z = 0$$

in the open unit disc $\{z : |z| < 1\}$. *[6]*

(b) Determine

$$\max\{|z\exp(iz^3 - 2)| : |z| \le 3\},$$

and find all points at which the maximum is attained, giving your answers in Cartesian form. *[10]*

[END OF QUESTION PAPER]